

# HOMEOTOPY GROUPS OF ROOTED TREE LIKE NON-SINGULAR FOLIATIONS ON THE PLANE

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**ABSTRACT.** Let  $F$  be a non-singular foliation on the plane with all leaves being closed subsets,  $H^+(F)$  be the group of homeomorphisms of the plane which maps leaves onto leaves endowed with compact open topology, and  $H_0^+(F)$  be the identity path component of  $H^+(F)$ . The quotient  $\pi_0 H^+(F) = H^+(F)/H_0^+(F)$  is an analogue of a mapping class group for foliated homeomorphisms. We will describe the algebraic structure of  $\pi_0 H^+(F)$  under an assumption that the corresponding space of leaves of  $F$  has a structure similar to a rooted tree of finite diameter.

## 1. INTRODUCTION

Non-singular foliations on the plane were studied by W. Kaplan [5, 6] and H. Whitney [17] in the 40–50 years of the XX century. In particular, W. Kaplan in [6] has generalized a theorem of E. Kamke and proved that every non-singular foliation  $F$  on the plane admits a continuous function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  such that

- 1) the leaves of  $f$  are connected components of level sets  $f^{-1}(c)$ ,  $c \in \mathbb{R}$ ;
- 2) near each  $z \in \mathbb{R}^2$  there are local coordinates  $(u, v)$  in which  $f(u, v) = u + f(z)$ .

This result was further extended to foliations with singularities by W. Boothby [2], and J. Jenkins and M. Morse [4]. Topological classification of different kinds of functions on surfaces was investigated in many papers, see e.g. A. Fomenko and A. Bolsinov [1], A. Oshemkov [9], V. Sharko [14], [15], O. Prishlyak [12], [13], E. Polulyakh and I. Yurchuk [10], E. Polulyakh [11], V. Sharko and Yu. Soroka [16].

W. Kaplan in [5, 6] has also mentioned that a non-singular foliation on the plane is glued of countably many strips along open boundary intervals and such that each strip has a foliation by parallel lines. In a recent paper S. Maksymenko and E. Polulyah [8] studied non-singular foliations  $F$  on arbitrary non-compact surfaces  $\Sigma$  glued from strips in a similar way. They proved contractibility of the connected components of groups  $H(F)$  of homeomorphisms of  $\Sigma$  mapping leaves onto leaves. Thus the homotopy type of  $H(F)$  is determined by the quotient group  $\pi_0 H(F) = H(F)/H_0(F)$  of path components of  $H(F)$ , where  $H_0(F)$  is the identity path component of  $H(F)$ .

In the present paper we compute the groups  $\pi_0 H(F)$  for a special class of non-singular foliations on the plane whose spaces of leaves have the structure similar to rooted trees of finite diameter, see Theorem 4.5.

## 2. STRIPED SURFACES

Let  $\Sigma_i$  be a surface with a foliation  $F_i$ ,  $i = 1, 2$ . Then a homeomorphism  $h : \Sigma_1 \rightarrow \Sigma_2$  will be called *foliated* if it maps leaves of  $F_1$  onto leaves of  $F_2$ .

**Definition 2.1.** A subset  $S \subset \mathbb{R}^2$  will be called a *model strip* if the following two conditions hold:

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- 1)  $\mathbb{R} \times (-1, 1) \subseteq S \subset \mathbb{R} \times [-1, 1]$ ;
- 2)  $S \cap \mathbb{R} \times \{-1, 1\}$  is a union of open mutually disjoint finite intervals.

Put

$$\partial_- S = S \cap (\mathbb{R} \times \{-1\}), \quad \partial_+ S = S \cap (\mathbb{R} \times \{1\}), \quad \partial S = \partial_- S \cup \partial_+ S.$$

Notice that every model strip has an oriented foliation consisting of horizontal arcs  $\mathbb{R} \times t$ ,  $t \in (-1, 1)$ , and connected components of  $\partial S$ .

Let  $\{S_\lambda\}_{\lambda \in \Lambda}$  be an arbitrary family of model strips, and

$$X = \bigcup_{\lambda \in \Lambda} \partial_- S_\lambda, \quad Y = \bigcup_{\lambda \in \Lambda} \partial_+ S_\lambda.$$

By Definition 2.1,  $X$  and  $Y$  are disjoint unions of open intervals, therefore one can also write

$$X = \bigcup_{\alpha \in A} X_\alpha, \quad Y = \bigcup_{\beta \in B} Y_\beta,$$

where  $X_\alpha$  and  $Y_\beta$  are open boundary intervals of those models strips and  $A$  and  $B$  are some index sets.

We will now glue model strips  $S_\lambda$  by identifying some of the intervals of  $X_\alpha$  with some of the intervals of  $Y_\beta$ . In order to make this let us fix any set of indexes  $C$  and two injective maps  $p : C \rightarrow A$  and  $q : C \rightarrow B$ . Notice that for each  $\gamma \in C$  there exists a unique preserving orientation affine homeomorphism  $\varphi_\gamma : X_{p(\gamma)} \rightarrow Y_{q(\gamma)}$ . Then the quotient space

$$(2.1) \quad \Sigma := \bigsqcup_{\lambda \in \Lambda} S_\lambda / \{X_{p(\gamma)} \overset{\varphi_\gamma}{\sim} Y_{q(\gamma)}\}$$

will be called a *striped surface*.

*Remark 2.2.* A unique preserving orientation affine homeomorphism  $\phi : (a, b) \rightarrow (c, d)$  is given by  $\phi(t) = \frac{c-d}{b-a}(t-a)$ .

*Remark 2.3.* In [8] a wider class of striped surfaces is considered: it is also allowed to identify arbitrary connected components of  $\bigsqcup_{\lambda \in \Lambda} \partial S_\lambda$  and the gluing affine homeomorphisms may reverse orientation.

Let also  $p : \bigsqcup_{\lambda \in \Lambda} S_\lambda \rightarrow \Sigma$  be the quotient map and  $p_\lambda : S_\lambda \rightarrow \Sigma$  be the restriction of  $p$  to the model strip  $S_\lambda$ . Then the pair  $(S_\lambda, p_\lambda)$  will be called a *chart* for the strip  $S_\lambda$ .

Since the homeomorphism  $\varphi_\gamma$  identifies leaves of such foliations, we see that every striped surface has the foliation  $F$  consisting of foliations on model strips. This foliation will be called *canonical*.

Moreover, each leaf of the foliation on the model strip is oriented and the gluing preserves orientation. Therefore all leaves of  $F$  are oriented.

**Special leaves.** Let  $U \subset \Sigma$  be a subset. Then the union of all leaves of  $F$  intersecting  $U$  is called the *saturation* of  $U$  with respect to  $F$  and denoted by  $Sat(U)$ .

A leaf  $\omega$  of  $F$  will be called *special* if

$$\omega \neq \bigcap_{N(\omega)} \overline{Sat(N(\omega))},$$

where  $N(\omega)$  runs over all open neighborhoods of  $\omega$ .

For instance each leaf  $\omega$  belonging to the interior of a strip is non-special. Moreover, suppose  $\omega = X_{p(\gamma)} \sim Y_{q(\gamma)}$  is a leaf such that  $\partial_- S_\lambda = X_{p(\gamma)}$  and  $\partial_+ S_{\lambda'} = Y_{q(\gamma)}$ , see Figure 2.1(a). Then the topological structure of the foliation  $F$  near  $\omega$  is “similar” to the structure of  $F$  near “internal” leaves of strips and such a leaf is non-special as well, see [8, Lemma 3.2].

It also follows from that lemma that  $\omega$  is special if and only if one of the following two conditions hold, see Figure 2.1(b):

- 1)  $\omega$  is the image of gluing of leaves  $X_{p(\gamma)}$  and  $Y_{q(\gamma)}$  such that either  $X_{p(\gamma)} \subsetneq \partial_- S_\lambda$  or  $Y_{q(\gamma)} \subsetneq \partial_+ S_{\lambda'}$  for some  $\gamma \in C$ ,  $\lambda, \lambda' \in \Lambda$ ;
- 2)  $\omega \subsetneq \partial_- S_\lambda$  or  $\omega \subsetneq \partial_+ S_\lambda$  for some  $\lambda \in \Lambda$ .

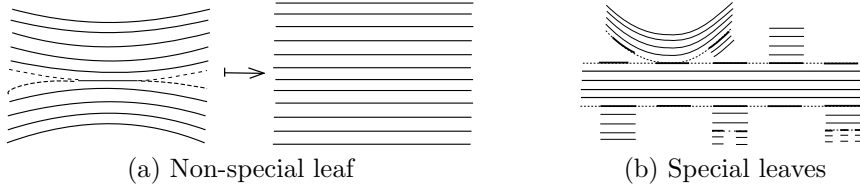


FIGURE 2.1.

**Reduced striped surfaces.** A striped surface  $\Sigma$  will be called *reduced* whenever a leaf  $\omega$  is special if and only if one of the following conditions holds:

- 1)  $\omega$  is an image of gluing of some leaves  $X_{p(\gamma)} \sim Y_{q(\gamma)}$  for some  $\gamma \in C$ ;
- 2)  $\omega \subsetneq \partial_- S_\lambda$  or  $\omega \subsetneq \partial_+ S_\lambda$  for some  $\lambda \in \Lambda$ .

Let  $S$  be a model strip such that  $\partial_- S = (0, 1) \times -1$  and  $\partial_+ S = (0, 1) \times 1$ . Let also  $\phi : \partial_- S \rightarrow \partial_+ S$  be a homeomorphism defined by  $\phi(t, -1) = (t, 1)$ ,  $t \in (0, 1)$ , and  $\mathcal{C} = S/\phi$  be the quotient space obtained by identifying each  $x \in \partial_- S$  with  $\phi(x) \in \partial_+ S$ .

Then  $\mathcal{C}$  is a striped surface homeomorphic with a cylinder, and its canonical foliation has no special leaves.

It follows from [8, Theorem 3.7] that every striped surface (in the sense of (2.1), see Remark 2.3) is foliated homeomorphic either to  $\mathcal{C}$  or to a reduced surface.

**Graph of a striped surface.** For a reduced striped surface  $\Sigma$  not foliated homeomorphic with  $\mathcal{C}$  define an oriented graph  $\Gamma(\Sigma)$  whose vertexes are strips and whose edges are special leaves. More precisely: if  $\omega = X_{p(\gamma)} \sim Y_{q(\gamma)}$  is a special leaf of  $F$ ,  $X_{p(\gamma)} \subset \partial_- S_{\lambda_0}$ , and  $Y_{q(\gamma)} \subset \partial_+ S_{\lambda_1}$ , then we assume that  $\omega$  is an *edge* between *vertices*  $S_{\lambda_0}$  and  $S_{\lambda_1}$  oriented from  $S_{\lambda_1}$  to  $S_{\lambda_0}$ .

If  $\lambda_0 = \lambda_1$ , then  $\omega$  correspond to a loop in  $\Gamma(\Sigma)$  at  $S_{\lambda_0} = S_{\lambda_1}$  being a vertex of  $\Gamma(\Sigma)$ .

Since a model strip may have infinitely many boundary components, we see that a graph  $\Gamma(\Sigma)$  can be not locally finite. On the other hand, it can have a finite diameter  $\text{diam } \Gamma(\Sigma)$ , being the minimal non-negative integer  $d$  such that every two vertices  $v_1$  and  $v_2$  are connected in  $\Gamma(\Sigma)$  by a path consisting at most  $d$  edges.

**Admissible striped surfaces.** Recall that a family  $\mathcal{V} = \{V_i\}_{i \in \Lambda}$  of subsets in a topological space  $X$  is called *locally finite* whenever for each  $x \in X$  there exists an open neighborhood intersecting only finitely many elements from  $\mathcal{V}$ .

It is well known and is easy to see that *a union of a locally finite family of closed subsets is closed*, e.g. [7, Chapter 1, § 5.VIII].

**Definition 2.4.** A model strip  $S$  will be called *admissible* if the closures of intervals in  $\partial_- S$  and  $\partial_+ S$  are mutually disjoint and constitute a locally finite family in  $\mathbb{R}^2$ .

*Example 2.5.* A model strip with

$$\partial_+ S = \bigcup_{n \in \mathbb{Z} \setminus \{-1, 0\}} \left( \frac{1}{n+1}, \frac{1}{n} \right) \times 1$$

is not admissible, since condition 2) of Definition 2.1 fails.

It will be convenient to use the following notation:

$$[0] = \emptyset, \quad [n] = \{1, 2, \dots, n\}, \quad -\mathbb{N} = \{-1, -2, \dots\}.$$

Let also  $J_i = (i, i + 0.5)$ ,  $i \in \mathbb{Z}$ , and for a subset  $\Delta \subset \mathbb{Z}$  denote

$$A_\Delta = \bigcup_{i \in \Delta} J_i.$$

In particular, consider the following collections of mutually disjoint open intervals:

$$\begin{aligned} A_{[n]} &= \bigcup_{i=1}^n (i, i + 0.5), & n = 0, 1, \dots, & & A_{\mathbb{N}} &= \bigcup_{i \in \mathbb{N}} (i, i + 0.5), \\ A_{-\mathbb{N}} &= \bigcup_{-i \in \mathbb{N}} (i, i + 0.5), & & & A_{\mathbb{Z}} &= \bigcup_{i \in \mathbb{Z}} (i, i + 0.5), \end{aligned}$$

which will be called *standard*. The following easy lemma is left for the reader.

**Lemma 2.6.** *Let  $S$  be an admissible model strip. Then there exists a homeomorphism  $h : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  preserving each line  $\mathbb{R} \times t$ ,  $t \in (-1, 1)$ , with its orientation, and such that  $h(S)$  is a model strip with  $\partial_- h(S) = A_\alpha \times \{-1\}$  and  $\partial_+ h(S) = A_\beta \times \{1\}$ , where  $A_\alpha$  and  $A_\beta$  are standard collections of intervals, i.e.  $\alpha, \beta \in \{[0], [1], \dots, \mathbb{N}, -\mathbb{N}, \mathbb{Z}\}$ , see Figure 2.2. Moreover,  $\alpha$  and  $\beta$  do not depend on a particular choice of such  $h$ .*

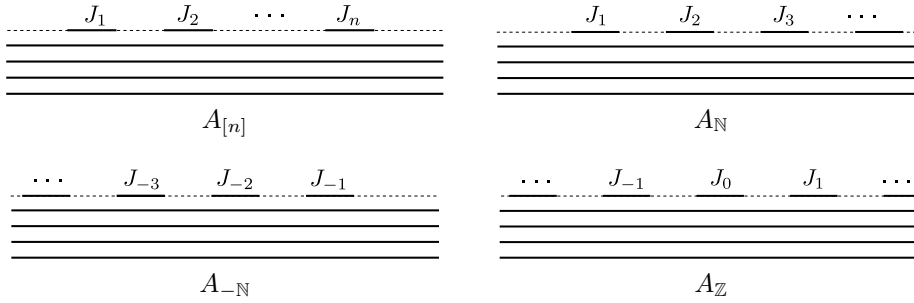


FIGURE 2.2. Types of  $\partial_+ S$

Thus for an admissible model strip  $S$  its foliated topological type is determined by the ordinal type of collections of boundary intervals in  $\partial_- S$  and  $\partial_+ S$ .

### 3. WREATH PRODUCTS

Let  $H$  and  $S$  be two groups. Denote by  $\text{Map}(H, S)$  the group of all *maps* (not necessarily homomorphisms)  $\varphi : H \rightarrow S$  with respect to the point-wise multiplication. Then the group  $H$  acts on  $\text{Map}(H, S)$  by the following rule: the result of the action of  $\varphi \in \text{Map}(H, S)$  on  $h \in H$  is the composition map:

$$\varphi \circ h : H \longrightarrow H \longrightarrow S.$$

The semidirect product  $\text{Map}(H, S) \rtimes H$  corresponding to this action will be denoted by  $S \wr H$  and called the *wreath product* of  $S$  and  $H$ . Thus

$$S \wr H = \text{Map}(H, S) \rtimes H$$

is the Cartesian product  $\text{Map}(H, S) \times H$  with the multiplication given by the formula

$$(\varphi_1, h_1) \cdot (\varphi_2, h_2) = ((\varphi_1 \circ h_2) \cdot \varphi_2, h_1 \cdot h_2)$$

for  $(\varphi_1, h_1), (\varphi_2, h_2) \in \text{Map}(H, S) \rtimes H$ .

Let  $\varepsilon : H \rightarrow S$  be the constant map into the unit of  $S$ . Then the pair  $(\varepsilon, \text{id}_H)$  is the unit element of  $S \wr H$ . Moreover, if  $(\varphi, h) \in S \wr H$  and  $\varphi^{-1} \in \text{Map}(H, S)$  is the point-wise inverse of  $\varphi$ , then  $(\varphi^{-1} \circ h^{-1}, h^{-1})$  is the inverse of  $(\varphi, h)$  in  $S \wr H$ .

We also have the following exact sequence:

$$1 \rightarrow \text{Map}(H, S) \xrightarrow{i} S \wr H \xrightarrow{\pi} H \rightarrow 1,$$

where  $i(\varphi) = (\varphi, e)$ ,  $e$  is the unit of  $H$ , and  $\pi(\varphi, h) = h$ . Moreover,  $\pi$  admits a section  $s : H \rightarrow S \wr H$  defined by  $s(h) = (\varepsilon, h)$ .

#### 4. MAIN RESULT

**Homeotopy group of a canonical foliation.** Let  $\Sigma$  be striped surface with a canonical foliation  $F$ . Denote by  $H(F)$  the groups of all foliated homeomorphisms  $h : \Sigma \rightarrow \Sigma$ , i.e. homeomorphisms mapping leaves of  $F$  onto leaves. We will endow  $H(F)$  with the corresponding compact open topology.

Recall that all leaves of  $F$  are oriented. Then we denote by  $H^+(F)$  the subgroup of  $H(F)$  consisting of homeomorphisms  $h : \Sigma \rightarrow \Sigma$  such that for each leaf  $\omega$  the restriction map  $h : \omega \rightarrow h(\omega)$  is orientation preserving.

Let  $H_0^+(F)$  be the identity path component of  $H^+(F)$ . It consists of all  $h \in H^+(F)$  isotopic to  $\text{id}_\Sigma$  in  $H^+(F)$ . Then  $H_0^+(F)$  is a normal subgroup of  $H^+(F)$ , and the corresponding quotient

$$\pi_0 H^+(F) = H^+(F) / H_0^+(F)$$

will be called the *homeotopy* group of  $F$ .

**Class  $\mathfrak{F}$ .** Denote by  $\mathfrak{F}$  the class of striped surfaces

$$\Sigma = \bigsqcup_{\lambda \in \Lambda} S_\lambda / \{X_{p(\gamma)} \stackrel{\varphi_\gamma}{\sim} Y_{q(\gamma)}\}$$

satisfying the following conditions:

- 1) each  $S_\lambda$ ,  $\lambda \in \Lambda$ , is admissible,

$$\partial_- S_\lambda = J_1 \times \{-1\} = A_{[1]} \times \{-1\}, \quad \partial_+ S_\lambda = A_{\Delta_\lambda} \times \{1\},$$

where  $\Delta_\lambda$  coincides with one of the standard collections  $A_{[n]}$ ,  $A_{\mathbb{N}}$ ,  $A_{-\mathbb{N}}$ , or  $A_{\mathbb{Z}}$ ;

- 2) the graph  $\Gamma(\Sigma)$  is connected and has a finite diameter and no cycles.

In particular, if  $\Sigma \in \mathfrak{F}$ , then each model strip  $S_\lambda$  of  $\Sigma$  regarded as a vertex of  $\Gamma(\Sigma)$  has at most one incoming edge and at most countably many outgoing edges linearly ordered with respect to  $\Delta_\lambda$ .

Since  $\Gamma(\Sigma)$  is connected and has a finite diameter and no cycles, it follows that there exists a unique vertex having no incoming edges. We will call this vertex a *root* and the corresponding strip a *root* strip.

Thus every surface  $\Sigma \in \mathfrak{F}$  of diameter  $d$  can be represented as follows, see Figure 4.1:

$$(4.1) \quad \Sigma = S \cup_{\partial_+ S} \left( \bigcup_{i \in \Delta} \Sigma_i \right),$$

where

- $S$  is a root strip of  $\Sigma$ ,

$$\partial_- S = J_1 \times \{-1\}, \quad \partial_+ S = \bigcup_{i \in \Delta} J_i \times \{1\},$$

where  $\Delta \in \{[0], [1], \dots, \mathbb{N}, -\mathbb{N}, \mathbb{Z}\}$ .

- $\Sigma_i$  is either empty or it is a striped surface belonging to  $\mathfrak{F}$  and its graph  $\Gamma(\Sigma_i)$  has diameter less than  $d$ .

- Suppose  $\Sigma_i$  is non-empty and let  $S_i$  be the root strip of  $\Sigma_i$ . Then  $\partial_- S_i = J_1 \times \{-1\}$  is glued to the boundary interval  $J_i \times \{1\}$  of  $\partial_+ S$  by the homeomorphism

$$\varphi : J_1 \equiv (1, 1.5) \longrightarrow J_i \equiv (i, i + 0.5), \quad \varphi(t) = t + i - 1.$$

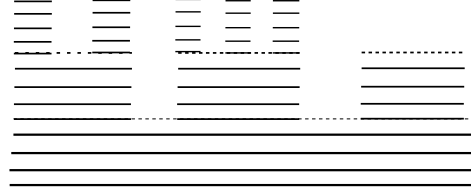


FIGURE 4.1. A striped surface  $\Sigma \in \mathfrak{F}$  whose graph  $\Gamma(\Sigma)$  has diameter 3

Obviously,  $\Sigma \in \mathfrak{F}$  is a connected and simply connected non-compact surface. Therefore it follows from [3] that the interior of  $\Sigma$  is homeomorphic to  $\mathbb{R}^2$ .

The class of homeotopy groups of foliations on striped surfaces which belongs to the class  $\mathfrak{F}$  will be denoted by  $\mathcal{P}$ , i.e.

$$\mathcal{P} = \{\pi_0 H^+(F) \mid F \text{ is a canonical foliation of some striped surface } \Sigma \in \mathfrak{F}\}.$$

We will also define another class of groups  $\mathcal{G}$ .

**Definition 4.1.** Let  $\mathcal{G}$  be the minimal class of groups satisfying the following conditions:

- 1)  $\{1\} \in \mathcal{G}$ ;
- 2) if  $A_i \in \mathcal{G}$  for  $i \in \mathbb{N}$ , then  $\prod_{i \in \mathbb{N}} A_i \in \mathcal{G}$ ;
- 3) if  $A \in \mathcal{G}$ , then  $A \wr \mathbb{Z} \in \mathcal{G}$ .

**Lemma 4.2.** A group  $G$  belongs to  $\mathcal{G}$  if and only if it can be obtained from the unit group  $\{1\}$  by a composition of finitely many operations of the following types:

- (a) countable direct products;
- (b) wreath product with the group  $\mathbb{Z}$ .

*Proof.* Let  $\mathcal{G}_0$  be the class of groups  $G$  which can be obtained from the unit group  $\{1\}$  by a composition of finitely many operations of types (a) and (b). Then any class of groups satisfying conditions 1)–3) of Definition 4.1 contains  $\mathcal{G}_0$ , whence  $\mathcal{G}_0 \subset \mathcal{G}$ . On the other hand,  $\mathcal{G}_0$  also satisfies conditions 1)–3) of Definition 4.1, whence  $\mathcal{G} \subset \mathcal{G}_0$  as well.  $\square$

Every representation  $\xi(G)$  of  $G$  as a composition of operations (a) and (b) will be called a *representation of  $G$  in the class  $\mathcal{G}$* . Such a representation is not unique. For example,

$$(4.2) \quad \mathbb{Z} \cong \{1\} \wr \mathbb{Z} \cong 1 \times (1 \wr \mathbb{Z}) \cong (1 \times 1 \times 1) \wr \mathbb{Z}.$$

**Definition 4.3.** The *height* of a representation  $\xi(G)$  of  $G$  in the class  $\mathcal{G}$  is a non-negative integer defined inductively as follows:

- 1)  $h(\{1\}) = 0$ ;
- 2)  $h(\xi(G) \wr \mathbb{Z}) = 1 + h(\xi(G))$ ;
- 3)  $h\left(\prod_{i \in \Lambda} \xi(A_i)\right) = 1 + \max_i \{h(\xi(A_i))\}$ .

*Example 4.4.* Below are examples of representations of groups  $\{1\}$ ,  $\mathbb{Z}$  and  $\mathbb{Z} \wr \mathbb{Z}$  in the class  $\mathcal{G}$  and their heights:

$$\begin{aligned} h(\{1\}) &= 0, & h(\{1\} \times \{1\}) &= 1, \\ h(\{1\} \wr \mathbb{Z}) &= 1, & h((\{1\} \times \{1\}) \wr \mathbb{Z}) &= 2, \\ h((\{1\} \wr \mathbb{Z}) \times (\{1\} \wr \mathbb{Z})) &= 2, & h(((\{1\} \times \{1\}) \wr \mathbb{Z}) \times (\{1\} \wr \mathbb{Z})) &= 3. \end{aligned}$$

Let  $\mathcal{G}' \subset \mathcal{G}$  be a subclass of  $\mathcal{G}$  consisting of groups admitting a representation of finite height in  $\mathcal{G}$ . The aim of the present paper is to prove the following theorem:

**Theorem 4.5.** *Classes  $\mathcal{P}$  and  $\mathcal{G}'$  coincide.*

In other words, a group  $G$  is isomorphic with a homeotopy group  $H^+(F)$  of some striped surface  $\Sigma \in \mathfrak{F}$  with a canonical foliation  $F$  if and only if  $G$  can be obtained from the unit group  $\{1\}$  by a composition of finitely many operations of types (a) and (b) of Lemma 4.2.

## 5. PRELIMINARIES

Let  $\Sigma$  be a striped surface belonging to  $\mathfrak{F}$  presented in the form (4.1), and  $S$  be the root strip of  $\Sigma$ . We will use coordinates  $(x, y)$  from the chart for  $S$ , so we can assume that  $\partial_+ S = \cup_{i \in \Delta} J_i \times \{1\}$ .

Notice that if  $h \in H^+(F)$ , then  $h(S) = S$ , whence there exists a unique number  $\eta(h) \in \mathbb{Z}$  such that in the chart for  $S$  we have that

$$h(J_i \times \{1\}) = J_{i+\eta(h)} \times \{1\}$$

for all  $i \in \Delta$ . One can easily check that the correspondence  $h \mapsto \eta(h)$  is a homomorphism

$$(5.1) \quad \eta : H^+(F) \rightarrow \mathbb{Z}.$$

Obviously,  $\eta$  can be a non-zero homomorphism only when  $\Delta = \mathbb{Z}$ .

Consider the following two subgroups of  $H^+(F)$ :

$$\begin{aligned} Q_S &= \{h \in H^+(F) \mid h(\omega) = \omega, \text{ for each leaf } \omega \text{ of } F \subset S\}, \\ H^+(F, S) &= \{h \in H^+(F) \mid h|_S = \text{id}|_S\}. \end{aligned}$$

It is evident that

$$(5.2) \quad H^+(F, S) \subset Q_S \subset \ker(\eta).$$

**Lemma 5.1.** *Embeddings (5.2) are homotopy equivalences.*

*Proof.* First we will construct a deformation of  $\ker(\eta)$  into  $Q_S$ . Let  $h \in \ker(\eta)$ . Since  $h(S) = S$ , it follows that  $h$  interchanges leaves of  $F$ . In the coordinates  $(x, y)$  in the chart for  $S$  these leaves are the lines  $y = \text{const}$ , whence

$$h(x, y) = (\alpha(x, y), \beta(y)),$$

where  $\alpha : S \rightarrow \mathbb{R}$  and  $\beta : [-1, 1] \rightarrow [-1, 1]$  are continuous functions such that for each  $y \in (0, 1)$  the correspondence  $x \mapsto \alpha(x, y)$  is a preserving orientation homeomorphism  $\mathbb{R} \rightarrow \mathbb{R}$ .

Then  $h \in Q_S$  iff  $\beta(y) = y$  for all  $y \in [0, 1]$ . Define the map  $H : \ker(\eta) \times [0, 1] \rightarrow \ker(\eta)$  by the formula

$$H(h, t)(z) = \begin{cases} (\alpha(x, y), (1-t)\beta(y) + ty), & z = (x, y) \in S, \\ z, & z \in \Sigma \setminus S. \end{cases}$$

One can easily check that  $H_0 = \text{id}_{\ker(\eta)}$ ,  $H_t(Q_S) \subset Q_S$  for all  $t \in [0, 1]$ , and  $H(h, 1) \in Q_S$ . Hence  $H$  is a deformation of  $\ker(\eta)$  into  $Q_S$ , and so the inclusion  $Q_S \subset \ker(\eta)$  is a homotopy equivalence.

Similarly, let  $h \in Q_S$ , so

$$h(x, y) = (\alpha(x, y), y)$$

for all  $(x, y) \in S$ . Notice that  $h \in H^+(F, S)$  iff  $\alpha(x, y) = x$  and  $\beta(y) = y$  for all  $(x, y) \in S$ .

Let

$$h(x, y) = (\alpha_i(x, y), \beta_i(y))$$

be the restriction of  $h$  onto root strip  $S_i$  of  $\Sigma_i$  in the corresponding chart of  $S_i$ . Since  $\partial_- S_i = J_1 \times \{-1\}$ , we see that if  $h \in H^+(F, S)$ , then  $\alpha_i(x, -1) = x$  for all  $x \in J_1$  and  $i \in \Delta$ .

Fix a continuous function  $\varepsilon : [-1, 1] \rightarrow [0, 1]$  such that

$$\varepsilon(y) = \begin{cases} 0, & y \in (-1, -0.8), \\ 1, & y \in (0, 1) \end{cases}$$

and define the following homotopy  $G : Q_S \times [0, 1] \rightarrow Q_S$  by

$$G(h, t)(z) = \begin{cases} ((1-t)\alpha(x, y) + tx, y), & z = (x, y) \in S, \\ ((1-t\varepsilon(y))\alpha_i(x, y) + t\varepsilon(y)x, \beta(y)), & z = (x, y) \in S_i, \\ z & z \notin S \cup (\cup_{i \in \Delta} S_i). \end{cases}$$

Since  $\partial_- S_i$  is glued to the boundary component  $J_i \times \{1\}$  by an affine homeomorphism, and the formulas for  $G$  are affine for each fixed  $t$  and  $y$ , it follows that those formulas agree on  $J_i \times \{1\}$  and  $\partial_- S_i$ , c.f. [8]. This implies that  $G$  is a continuous map.

Moreover, one can easily check that  $G_0 = \text{id}_{Q_S}$ ,  $G_t(H^+(F, S)) \subset H^+(F, S)$  for all  $t \in [0, 1]$ , and  $G_1(Q_S) \subset H^+(F, S)$ . Hence  $G$  is a deformation of  $Q_S$  into  $H^+(F, S)$ , and therefore the inclusion  $H^+(F, S) \subset Q_S$  is a homotopy equivalence as well.  $\square$

Suppose  $\Sigma_i$  is non-empty for some  $i \in \Delta$ . Let  $F_i$  be the canonical foliation on  $\Sigma_i$  and  $S_i$  be the root strip of  $\Sigma_i$ . We will denote by  $H^+(F_i, \partial_- S_i)$  the subgroup of  $H^+(F_i)$  consisting of homeomorphisms fixed on  $\partial_- S_i$ .

If  $\Sigma_i = \emptyset$ , then we will assume that  $H^+(F_i, \partial_- S_i) = \{1\}$ .

**Lemma 5.2.** *We have an isomorphism*

$$\pi_0 \ker(\eta) \cong \prod_{i \in \Delta} \pi_0 H^+(F_i, \partial_- S_i).$$

*Proof.* Evidently, we have a canonical isomorphism

$$\alpha : H^+(F, S) \cong \prod_{i \in \Delta} H^+(F_i, \partial_- S_i), \quad \alpha(h) = (h|_{\Sigma_i})_{i \in \Delta}.$$

Then from Lemma 5.1 we get the following sequence of isomorphisms:

$$\pi_0 \ker(\eta) \cong \pi_0 H^+(F, S) \cong \pi_0 \prod_{i \in \Delta} H^+(F_i, \partial_- S_i) = \prod_{i \in \Delta} \pi_0 H^+(F_i, \partial_- S_i).$$

Lemma is proved.  $\square$

**Theorem 5.3.** 1) *If  $\eta$  is zero homomorphism, then the group  $\pi_0 H^+(F)$  is isomorphic to  $\prod_{i \in \Delta} \pi_0 H^+(F_i, \partial_- S_i)$ .*

2) *Suppose the image of  $\eta$  is  $k\mathbb{Z}$  for some  $k \geq 1$ , so  $\Delta = \mathbb{Z}$ . Then the group  $\pi_0 H^+(F)$  is isomorphic to  $\left( \prod_{i=0}^{k-1} \pi_0 H^+(F_i, \partial_- S_i) \right) \wr \mathbb{Z}$ .*



*Proof.* 1) The assumption that  $\eta$  is zero homomorphism means that  $H^+(F) = \ker(\eta)$ , whence we get from Lemma 5.2 that

$$\pi_0 H^+(F) \cong \prod_{i \in \Delta} \pi_0 H^+(F_i, \partial_- S_i).$$

2) Suppose  $\text{Im } \eta = k\mathbb{Z}$ . Then we have an *epimorphism*  $\hat{\eta} : H^+(F) \rightarrow \mathbb{Z}$  defined by  $\hat{\eta}(h) = \eta(h)/k$  and such that

$$h(\Sigma_r) = \Sigma_{r+k \cdot \hat{\eta}(h)}, \quad r = 0, 1, \dots, k-1.$$

Let

$$X = \bigcup_{i=0}^{k-1} \Sigma_i, \quad \partial_- X = \bigcup_{i=0}^{k-1} \partial_- S_i,$$

and  $F_X$  be the oriented foliation on  $X$  induced by  $F$ . Denote by  $H^+(F_X, \partial_- X)$  the group of homeomorphisms of  $X$  fixed on  $\partial_- X$  and mapping leaves of  $F_X$  onto leaves and preserving their orientation. Then we have a natural isomorphism

$$\prod_{i=0}^{k-1} H^+(F_i, \partial_- S_i) \cong H^+(F_X, \partial_- X)$$

which yields an isomorphism

$$\prod_{i=0}^{k-1} \pi_0 H^+(F_i, \partial_- S_i) \cong \pi_0 H^+(F_X, \partial_- X).$$

Therefore for the proof of Theorem 5.3 we should construct an isomorphism

$$\beta : \pi_0 H^+(F) \longrightarrow \pi_0 H^+(F_X, \partial_- X) \wr \mathbb{Z} \equiv \text{Map}(\mathbb{Z}, \pi_0 H^+(F_X, \partial_- X)) \rtimes \mathbb{Z}.$$

Fix any  $g \in H^+(F)$  with  $\hat{\eta}(g) = 1$ . Then

$$g^{-\hat{\eta}(h)} \circ h(\Sigma_i) = \Sigma_i,$$

for all  $h \in H^+(F)$  and  $i \in \mathbb{Z}$ , whence  $g^{-\hat{\eta}(h)} \circ h \in \ker(\eta)$ . Thus we get a well-defined function

$$\varphi_h : \mathbb{Z} \rightarrow \pi_0 H^+(F_X, \partial_- X), \quad \varphi_h(j) = \left[ g^{-j-\hat{\eta}(h)} \circ h \circ g^j \right]_X.$$

Define the following map:

$$\beta : \pi_0 H^+(F) \longrightarrow \pi_0 H^+(F_X, \partial_- X)$$

by the formula

$$\beta(h) = (\varphi_h, \hat{\eta}(h)), \quad h \in \pi_0 H^+(F).$$

We claim that  $\beta$  is an isomorphism. First notice that the composition operation in  $H^+(F_X, \partial_- X) \wr \mathbb{Z}$  is given by the following rule:

$$(\varphi_{h_1}, n) \cdot (\varphi_{h_2}, m) = (\varphi_{h_1}^m \cdot \varphi_{h_2}, n + m),$$

where  $\varphi_h^m(j) = \varphi_h(j + m)$ .

**Proof that  $\beta$  is a homomorphism.** Let  $h_1, h_2 \in H^+(F)$ . Then

$$\begin{aligned} \beta(h_1) \circ \beta(h_2) &= (\varphi_{h_1}, \hat{\eta}(h_1)) \cdot (\varphi_{h_2}, \hat{\eta}(h_2)) \\ &= (\varphi_{h_1}^{\hat{\eta}(h_2)} \cdot \varphi_{h_2}, \hat{\eta}(h_1) + \hat{\eta}(h_2)) \\ &= \left( [g^{-j-\hat{\eta}(h_1)-\hat{\eta}(h_2)} \circ h_1 \circ g^{j+\hat{\eta}(h_2)} \circ g^{-j-\hat{\eta}(h_2)} \circ h_2 \circ g^j]_X, \hat{\eta}(h_1 \circ h_2) \right) \\ &= \left( [g^{-j-\hat{\eta}(h_1 \circ h_2)} \circ h_1 \circ h_2 \circ g^j]_X, \hat{\eta}(h_1 \circ h_2) \right) \\ &= (\varphi_{h_1 \circ h_2}, \hat{\eta}(h_1 \circ h_2)) = \beta(h_1 \circ h_2). \end{aligned}$$

**Proof that  $\beta$  is injective.** Let  $h \in H^+(F)$  be such that  $[h] \in \ker \beta$ . We should prove that  $h$  is isotopic in  $H^+(F)$  to  $\text{id}_\Sigma$ .

The assumption  $[h] \in \ker \beta$  means that  $\beta(h) = (\varphi_h, \hat{\eta}(h)) = (\varepsilon, 0)$ , where  $\varepsilon : \mathbb{Z} \rightarrow [\text{id}_X]$  is the constant map into the unit of  $\pi_0 H^+(F_X, \partial_- X)$ . In particular, since  $\eta(h) = 0$ , we get from Lemma 5.1 that  $h$  is isotopic in  $H^+(F)$  to a homeomorphism fixed on  $S$ . Therefore we can assume that  $h$  itself is fixed on  $S$ , that is  $h \in H^+(F, S)$ . Then

$$(5.3) \quad \varphi_h(j) = [g^{-j} \circ h \circ g^j|_X] = \varepsilon(j) = [\text{id}_X] \in \pi_0 H^+(F_X, \partial_- X)$$

for each  $j \in \mathbb{Z}$ . In other words,  $g^{-j} \circ h \circ g^j|_X$  is isotopic to  $\text{id}_X$  relatively  $\partial_- X$ .

It suffices to prove that for each  $i \in \mathbb{Z}$  the restriction  $h|_{\Sigma_i}$  is isotopic in  $H^+(F_i, \partial_- S_i)$  to  $\text{id}_{\Sigma_i}$  relatively to  $\partial_- S_i$ .

Write  $i = r + jk$  for a unique  $r \in \{0, k-1\}$ . Then we have the following commutative diagram:

Therefore, we get from (5.3) that  $[h|_{\Sigma_i}] = [\text{id}_{\Sigma_i}] \in H^+(F_i, \partial_- S_i)$ . Hence  $h$  is isotopic to  $\text{id}_\Sigma$  in  $H^+(F)$ .

**Proof that  $\beta$  is surjective.** Let  $(\varphi, n) \in \pi_0 H^+(F_X, \partial_- X) \wr \mathbb{Z}$ . For each  $j \in \mathbb{Z}$  fix a homeomorphism  $h_j \in H^+(F_X, \partial_- X)$  such that  $[h_j] = \phi(j) \in \pi_0 H^+(F_X, \partial_- X)$ . Now define the following homeomorphism  $\hat{h}$  of  $\Sigma$  by the formula:

$$\hat{h} = \begin{cases} \text{id}_S, & \text{on } S, \\ [g^j \circ h_j \circ g^{-j}] & \text{on } g^j(X) \end{cases}$$

and put  $h = g^n \circ \hat{h}$ . Then it is easy to see that  $\beta([h]) = (\phi, n)$ , whence  $\beta$  is surjective. Thus  $\beta$  is an isomorphism.  $\square$

## 6. PROOF OF THEOREM 4.5

We should prove that  $\mathcal{P} = \mathcal{G}'$ .

1. First we will show that  $\mathcal{G}' \subset \mathcal{P}$ .

Let  $G \in \mathcal{G}'$ , so  $G$  has a representation  $\xi(G)$  in the class  $\mathcal{G}$  of finite height  $k = h(\xi(G))$ . We have to show that there exists a striped surface  $\Sigma \in \mathfrak{F}$  with canonical foliation  $F$  such that  $G \cong \pi_0 H^+(F)$ .

If  $k = h(\xi(G)) = 0$ , then  $G$  is the unit group  $\{1\}$  and  $\xi(G) = \{1\}$ . Let  $S$  be an admissible model strip with  $\partial_- S = A_{[1]} \times \{-1\}$  and  $\partial_+ S = \emptyset$ . Then  $S \in \mathfrak{F}$ . Let also  $F$  be the canonical foliation on  $S$ . Then

$$\pi_0 H^+(F) = \{1\} = G,$$

i.e.  $G \in \mathcal{P}$ .

Suppose that we have established our statement for all  $k$  being less than some  $\bar{k} > 0$ . Let us prove it for  $k = \bar{k}$ . It follows from Definition 4.3 that either

- (i)  $\xi(G) = \prod_{i \in \mathbb{N}} A_i$  where each group  $A_i$  has a representation  $\xi(A_i)$  in the class  $\mathcal{G}$  of height  $h(\xi(A_i)) < k$ , or
- (ii)  $\xi(G) = A\mathbb{Z}$ , and  $A$  has a representation  $\xi(A)$  in the class  $\mathcal{G}$  of height  $h(\xi(A)) < k$ .

In the case (i) due to the inductive assumption for each  $i \in \mathbb{N}$  there exists a striped surface  $\Sigma_i \in \mathfrak{F}$  with foliations  $F_i$  such that  $A_i = \pi_0 H^+(F_i)$ .

Let  $S$  be an admissible model strip with  $\partial_- S = A_{[1]} \times \{-1\}$  and  $\partial_+ S = A_{\mathbb{N}} \times \{1\}$ , and  $S_i$  be the root strip of  $\Sigma_i$ ,  $i \in \mathbb{N}$ . Define the striped surface

$$\Sigma = S \cup_{\partial_+ S} \left( \bigcup_{i \in \mathbb{N}} \Sigma_i \right)$$

obtained by identifying  $\partial_- S_i \subset \Sigma_i$  with  $J_i \times \{1\} \subset \partial_+ S$ . Then by Theorem 5.3  $\eta$  is a trivial homomorphism, and  $\pi_0 H^+(F) \cong \prod_{i \in \mathbb{N}} \pi_0 H^+(F_i) \cong \prod_{i \in \mathbb{N}} A_i \cong G$ . So  $G \in \mathcal{P}$ .

In the case (ii) again by inductive assumption there exists a striped surface  $\widehat{\Sigma} \in \mathfrak{F}$  with a canonical foliation  $\widehat{F}$  such that  $A = \pi_0 H^+(\widehat{F})$ .

Take countably many copies  $\widehat{\Sigma}_i$ ,  $i \in \mathbb{Z}$ , of  $\widehat{\Sigma}$ . Let  $\widehat{S}_i$  be the root strip of  $\widehat{\Sigma}_i$  and  $\widehat{F}_i$  be the canonical foliation on  $\widehat{\Sigma}_i$ .

Let also  $S$  be an admissible model strip with  $\partial_- S = A_{[1]} \times \{-1\}$  and  $\partial_+ S = A_{\mathbb{Z}} \times \{1\}$ . Define the following striped surface:

$$\Sigma = S \cup_{\partial_+ S} \left( \bigcup_{i \in \mathbb{N}} \widehat{\Sigma}_i \right).$$

Obtained by gluing each  $\widehat{\Sigma}_i$  to  $S$  by identifying  $\partial_- \widehat{S}_i \subset \widehat{\Sigma}_i$  with  $J_i \times \{1\} \subset \partial_+ S$ ,  $i \in \mathbb{Z}$ .

Then for every pair  $i, j \in \mathbb{Z}$  there exists  $h \in H^+(F)$  such that  $h(\widehat{\Sigma}_i) = \widehat{\Sigma}_j$ , whence the homomorphism  $\eta$ , see (5.1) is surjective. Hence by Theorem 5.3

$$\pi_0 H^+(F) \cong \pi_0 H^+(\widehat{F}) \wr \mathbb{Z} \cong A \wr \mathbb{Z} \cong G.$$

Thus,  $G \in \mathcal{P}$  and so  $\mathcal{G}' \subset \mathcal{P}$ .

**2.** Conversely, let us show that  $\mathcal{P} \subset \mathcal{G}'$ .

Let  $\Sigma \in \mathfrak{F}$  be a striped surface presented in the form (2.1) with canonical foliation  $F$  and such  $\text{diam } \Gamma(\Sigma) = k$ . We should prove that  $\pi_0 H^+(F)$  has a finite presentation in the class  $\mathcal{G}$ , which means that  $\pi_0 H^+(F) \in \mathcal{G}'$ .

If  $k = 0$ , then  $\Sigma$  is an admissible model strip with

$$\partial_- \Sigma = A_{[1]} \times \{-1\}, \quad \partial_+ \Sigma = A_\alpha, \quad \alpha \in \{[0], [1], \dots, \mathbb{N}, -\mathbb{N}, \mathbb{Z}\}.$$

Then it easily follows from Theorem 5.3 that  $\pi_0 H^+(F) \cong \mathbb{Z} \cong \{1\} \wr \mathbb{Z}$  if  $\alpha = \mathbb{Z}$ , and  $\pi_0 H^+(F) \cong \{1\}$  otherwise. In both cases  $\pi_0 H^+(F) \in \mathcal{G}$ .

Suppose that we have established our statement for all  $k$  being less than some  $\bar{k} > 0$ . We should prove it for  $k = \bar{k}$ . Let

$$\Sigma = S \cup_{\partial_+ S} \left( \bigcup_{i \in \Delta} \Sigma_i \right) \in \mathfrak{F}$$

be such that  $\Gamma(\Sigma)$  has diameter  $k$ . Then  $\Gamma(\Sigma_i)$  has diameter less than  $k$ , and so by inductive assumption  $\pi_0 H^+(F_i, \partial_- S_i) \in \mathcal{G}$ . Moreover, according to Theorem 5.3 we have that

- (i) if  $\text{image}(\eta) = 0$ , then  $\pi_0 H^+(F) \cong \prod_{i \in \Delta} \pi_0 H^+(F_i, \partial_- S_i) \in \mathcal{G}$ ,
- (ii) if  $\text{image}(\eta) = k\mathbb{Z}$ , then  $\pi_0 H^+(F) \cong \left( \prod_{i=0}^{k-1} \pi_0 H^+(F_i, \partial_- S_i) \right) \wr \mathbb{Z} \in \mathcal{G}$ .

Thus  $\mathcal{P} \subset \mathcal{G}'$ , and so  $\mathcal{P} = \mathcal{G}'$ . Theorem 5.3 completed.

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